Q1 a) Find Laplace transform of f(t) = e⁻⁴ᵗ sin3t.cos2t.   **(5)**

$\sin 2t.\sin 3t = \frac{1}{2}. 2\sin 3t \sin 2t = \frac{-1}{2}[\cos 5t - \cos t]$

$\therefore L(\sin 2t.\sin 3t) = \frac{-1}{2}[L(\cos 5t) - L(\cos t)] = \frac{-1}{2}\left[\frac{s}{s^2+5^2} - \frac{s}{s^2+1^2}\right]$

Now, by shifting theorem,

$L(e^{-4t}\sin 2t.\sin 3t) = \frac{-1}{2}\left[\frac{s+4}{(s+4)^2+5^2} - \frac{s+4}{(s+4)^2+1^2}\right]$

$= \frac{-1}{2}\left[\frac{s+4}{s^2+8s+41} - \frac{s+4}{s^2+8s+17}\right]$

$= \frac{12(s+4)}{(s^2+8s+41)(s^2+8s+17)}$

Q1 b) Show that the set of functions f(x) = 1, g(x) =x are orthogonal on (-1,1). Determine the constants a and b such that the function h(x) = 1+ax+bx² is orthogonal to both f(x) and g(x).   **(5)**

$\int_{-1}^{1} f(x).g(x)dx = \int_{-1}^{1}(x)dx = 0$

Hence, the given sets are orthogonal on each other over (-1,1).

$\int_{-1}^{1} f(x).h(x)dx = 0 \quad \therefore \int_{-1}^{1} 1.(-1 + ax + bx^2)dx = 0$

$\therefore [-x + \frac{ax^2}{2} + \frac{bx^3}{3}] = 0 \qquad \therefore -2 + \frac{2b}{3} = 0 \quad \therefore b = 3$

$\int_{-1}^{1} g(x).h(x)dx = 0 \quad \therefore \int_{-1}^{1} x.(-1 + ax + bx^2)dx = 0$

$\therefore [\frac{-x^2}{2} + \frac{ax^3}{3} + \frac{bx^4}{4}] = 0 \qquad \therefore a = 0$

$\therefore h(x) = 3x^2 - 1$

Q1 c) Evaluate $\int (z^2 - 2\bar{z} + 1)\, dz$ where C is circle |z| = 1.   **(5)**

Let $z = re^{i\theta}$ $\therefore \bar{z} = re^{-i\theta}$ Here, r = 1

Putting the values in the equation,

$\int(z^2 - 2\bar{z} + 1)\,dz \qquad = \qquad \int(e^{i\theta})^2 - 2\,e^{-i\theta} + 1)\,ie^{i\theta}d\theta$

$\qquad \theta$ goes from 0 to 2π

$$= \int_0^{2\pi}(e^{2i\theta} - 2\,e^{-i\theta} + 1)\,ie^{i\theta}d\theta$$

$$= i\int_0^{2\pi}(e^{3i\theta} - 2\,e^0 + e^{i\theta})d\theta$$

$$= i\left[\frac{e^{3i\theta}}{3i} - 2\,\theta + \frac{e^{i\theta}}{i}\right]$$

$$= \left[\frac{e^{6i\pi}}{3} - 4i\pi + e^{i2\pi}\right] - \left[\frac{1}{3} + 1\right]$$

$$= \left[\quad \frac{1}{3}(\cos 6\pi + i\sin 6\pi) - 4i\pi + (\cos 2\pi + i\sin 2\pi)\right] - \frac{4}{3}$$

$$\therefore \int(z^2 - 2\bar{z} + 1)\,dz = -4i\pi$$

Q1 d) Compute the Spearman's Rank correlation coefficient R and Karl Pearson's correlation r from the following data, **(5)**

x	12	17	22	27	32
y	113	119	117	115	121

Sr.No	X	X	x^2	Y	y	y^2	xy	R_1	R_2	$(R1-R2)^2$
1	12	-10	100	117	-4	16	40	5	5	0
2	17	-5	25	119	2	4	-10	4	4	4
3	22	0	0	117	0	0	0	3	3	0
4	27	5	25	115	-2	4	-10	2	2	4
5	32	10	100	121	4	16	40	1	1	0
N = 5	110			585		40	60			8

$R = 1 - \frac{6\sum D^2}{N^3 - N} = \mathbf{0.6}$ $\qquad \bar{X} = \frac{110}{5} = 22$, $\bar{Y} = \frac{585}{5} = 117$

$r = \frac{\sum xy}{\sqrt{\sum x^2 \cdot \sum y^2}} = \mathbf{0.6}$

2.a) Using Laplace transform, evaluate $\int_0^\infty e^{-t} \int_0^t \frac{\sin u}{u} \, du \, dt$ **(6)**

$L(\sin u) = \frac{1}{(s^2+1)}$

$L\left(\frac{\sin u}{u}\right) = \int_s^\infty \frac{ds}{(s^2+1)} = \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s$

$\therefore L\left(\int_0^t \frac{\sin u}{u} \, du\right) = \frac{1}{s} \cot^{-1}s$

Now, $\int_0^\infty e^{-st} \int_0^t \frac{\sin u}{u} \, du \, dt = \frac{1}{s} \cot^{-1}s$

Put s =1,

$\therefore \int_0^\infty e^{-t} \int_0^t \frac{\sin u}{u} \, du \, dt = \cot^{-1}s = \frac{\pi}{4}$

2.b) Find an analytic function f(z) = u +iv, if

$$u = e^{-x}\{(x^2 - y^2)\cos y + 2xy\sin y\}$$ **(6)**

f(z) = u +iv

$u = e^{-x}\{(x^2 - y^2)\cos y + 2xy\sin y\}$

$u_x = \cos y[e^{-x}(2x) + (x^2 - y^2)e^{-x}] + \sin y[e^{-x}(2y) + 2xy\,e^{-x}]$

$u_y = e^{-x}[(x^2 - y^2)(-\sin y) - \cos y(-2y)] + e^{-x}[2xy\cos y - \sin y 2x]$

$f'(z) = u_x - iu_y$

$\quad = \cos y[e^{-x}(2x) + (x^2 - y^2)e^{-x}] + \sin y[e^{-x}(2y) + 2xy\,e^{-x}] - i\{e^{-x}[(x^2 - y^2)(-\sin y) - \cos y(-2y)] + e^{-x}[2xy\cos y - \sin y 2x]\}$

Put x=z and y= 0

$\therefore f'(z) = (2ze^{-z} + e^{-z}z^2)$

$\therefore f(z) = \int(2ze^{-z} + e^{-z}z^2)dz$

$\therefore \mathbf{f(z) = -4ze^{-z} - 3e^{-z} - z^2e^{-z}}$

2.c) Obtain Fourier series of f(x) = x² in (0,2π). Hence, deduce that-

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \ldots$$ **(8)**

Let c=0 and c + 2l = 2 π

∴ l = π

$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)dx \quad = \frac{1}{\pi}\int_0^{2\pi} x^2 dx \quad = \frac{8\pi^2}{3}$

$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos\frac{n\pi x}{\pi}dx$

Solving the above integral and applying the limits, we get

$a_n = \frac{4}{n^2}$

$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin\frac{n\pi x}{\pi}dx$

Solving the above integral and applying the limits, we get

$b_n = \frac{-4\pi}{n}$

In Fourier series,

$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

$\therefore x^2 = \frac{8\pi^2/3}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos\frac{n\pi x}{\pi} + \sum_{n=1}^{\infty}\frac{-4\pi}{n}\sin\frac{n\pi x}{\pi}$

$\therefore x^2 = \frac{4\pi^2}{3} + 4[\frac{\cos 1x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \ldots\ldots] - 4\pi[\frac{\sin 1x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \ldots\ldots]$

Deduction: Put x = π

$\therefore \pi^2 = \frac{4\pi^2}{3} + 4[\frac{\cos 1\pi}{1^2} + \frac{\cos 2\pi}{2^2} + \frac{\cos 3\pi}{3^2} + \ldots\ldots] - 4\pi[\frac{\sin 1\pi}{1} + \frac{\sin 2\pi}{2} + \frac{\sin 3\pi}{3} + \ldots]$

$\therefore \frac{-\pi^2}{3} = -4[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{-1}{3^2} + \ldots\ldots] - 0$

$\therefore \mathbf{\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2}\ldots\ldots}$

3.a) Using Bender- Schmidt method, solve $\frac{\partial^2 u}{\partial x^2}$ - $\frac{\partial u}{\partial t}$ = 0, subject to the conditions, u(0,t) = 0 , u(4,t) = 0, u(x,0) = $x^2(16 - x^2)$ taking h =1, for 3 minutes. **(6)**

a = 1   h = 1

k = a$\frac{h^2}{2}$ = 0.5

We divide x interval into 4 parts by taking h=1

We also divide time interval by taking k =0.3 up to 3

t_0 = 0, t_1 = 0.5, t_2 = 1, t_3 = 1.5, t_4 = 2, t_5 = 2.5, t_6 = 3

By data, u (0, t) = 0

Hence, for all x = 0 and t = 0, 0.5, 1, 1.5, 2, 2.5, 3

∴ u (0, t) = 0 for all t.

By data, u(4,t) = 0

Hence, for all x = 4 and t = 0, 0.5, 1, 1.5, 2, 2.5, 3

Now, u(x,0) = $x^2(16 - x^2)$

We now calculate u for t=0 and x= 1,2,3,4

When x= 0, t = 0,   u = $0^2(16 - 0^2)$ = 0

    x= 1, t = 0,   u = $1^2(16 - 1^2)$ = 15

    x= 2, t = 0,   u = $2^2(16 - 2^2)$ = 48

    x= 3, t = 0,   u = $3^2(16 - 3^2)$ = 63

    x= 4, t = 0,   u = $4^2(16 - 4^2)$ = 0

Now, c = $\frac{a+b}{2}$

t,x	0	1	2	3	4
0	0	15	48	63	0
0.5	0	24	39	24	0
1	0	19.5	24	19.5	0
1.5	0	12	19.5	12	0

2	0	9.75	6	9.75	0
2.5	0	3	9.75	3	0
3	0	4.875	3	4.875	0

3.b) Using convolution theorem, find the inverse Laplace transform of

$$F(s) = \frac{s^2+s}{(s^2+1)(s^2+2s+2)}$$ (6)

Let $\varphi_1 = \frac{s+1}{(s^2+2s+2)}$ and $\varphi_2 = \frac{s}{(s^2+1)}$

$L^{-1}\frac{s+1}{(s+1)^2+1} = e^{-u}\, L^{-1}\frac{s}{(s^2+1)} = e^{-u}\cos u, \quad L^{-1}\frac{s}{(s^2+1)} = \cos u$

$\therefore L^{-1}\left[\frac{s+1}{(s^2+2s+2)} \cdot \frac{s}{(s^2+1)}\right] = \int_0^t e^{-u}\cos u\,.\cos(t-u)\,du$

$$= \frac{1}{2}\int_0^t e^{-u}[\cos(2u-1)+\cos t]\,du$$

$$= \frac{1}{2}\left[\frac{1}{5}e^{-u}[-\cos(2u-t)+2\sin(2u-t)-e^{-u}\cos t\right]$$

$$= \frac{1}{10}\left[e^{-t}[(2\sin t - 6\cos t)+(2\sin t + 6\cos t)\right]$$

3.c) Using Residue theorem, evaluate

i) $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$ 　　ii) $\oint \frac{z^2}{(z+1)^2(z-2)}\,dz$, C: $|z| = 1.5$ (8)

i) $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$

$e^{i\theta} = z \quad\quad i\,e^{i\theta}d\theta = dz \quad\quad\quad \therefore \frac{dz}{iz} = d\theta$ and $\cos\theta = \frac{z+z^{-1}}{2}$

$\therefore I = \oint \frac{1}{[2+\frac{z^2+1}{2z}]} \cdot \frac{dz}{iz} \quad = \frac{4}{i}\oint \frac{z\,dz}{[z^2+4z+1]}$ where C is the circle $|z| = 1.5$

Now the poles of f(z) are given by $z^2+4z+1 = 0$

$\therefore z = -2\pm\sqrt{3}$

Both the poles are of order 2. But the pole $\alpha = -2 - \sqrt{3}$ lies outside the circle and the pole $\beta = -2 + \sqrt{3}$ lies inside the circle.

$\therefore$ Residue at $(z = \beta) = \lim\limits_{z \to \beta} \frac{1}{1!} \frac{d}{dz} [(z - \beta)^2 \cdot \frac{z}{(z-\beta)^2 (z-\alpha)^2}]$

Solving the above limit, we get

$$= - \frac{\beta + \alpha}{(\beta - \alpha)^3}$$

But, $\beta + \alpha = -4$ and $\beta - \alpha = 2\sqrt{3}$

$\therefore$ Residue (at $z = \beta$) $= \frac{4}{24\sqrt{3}} = \frac{1}{6\sqrt{3}}$

$\therefore I = 2\pi i \cdot \frac{4}{i} \cdot \frac{1}{6\sqrt{3}} = \frac{4\pi}{3\sqrt{3}}$

ii) $\oint \frac{z^2}{(z+1)^2 (z-2)} dz$, C: |z| = 1.5

Now, the poles are given by $(z + 1)^2 (z - 2) = 0$

$\therefore z = -1, 2$

The pole $z = -1$ lies inside the circle $|z| = 1.5$ and the pole $z = 2$ lies outside.

Now, the residue of f(z) (at z = -1) $= \lim\limits_{z \to -1} \frac{1}{1!} \frac{d}{dz} [(z + 1)^2 \cdot \frac{z^2}{(z+1)^2 (z-2)^2}]$

$$= \lim\limits_{z \to -1} \frac{d}{dz} [\frac{z^2}{(z-2)^2}]$$

$$= \lim\limits_{z \to -1} [\frac{(z-2)^2 \cdot 2z - z^2 \cdot 2(z-2)}{(z-2)^4}] = \lim\limits_{z \to -1} [\frac{-4z}{(z-2)^3}] = \frac{-4}{27}$$

$\therefore I = 2\pi i \cdot \frac{-4}{27} = \frac{-8\pi i}{27}$

4.a) Solve by Crank Nicholson simplified formula $\frac{\partial^2 u}{\partial x^2} - 16\frac{\partial u}{\partial t} = 0$, u(0,t) = 0, u(1,t) – 200t, u(x,0) = 0 taking h = 0.25 for one time step. **(6)**

a = 16 and h = 0.25

$k = ah^2 = 16 \times \frac{1}{16} = 1$

Now, $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$

By data,

When t=0, u=0, x = 0, 0.25, 0.5, 0.75, 1.00

When x =0, for all values of t, u=0 when t =0,1

When x =1 and t =0, u =0 and when x =1, t=1, u =200 x 1 =200.

t,x	0.00	0.25	0.50	0.75	1.00
0	0	0	0	0	0
1	0	u_1	u_2	u_3	200

$e = \frac{1}{4}(a+b+c+d)$

$u_1 = \frac{1}{4}(0+0+0+u_2) = u_2/4$

$u_2 = \frac{1}{4}(0+0+u_1+u_3) = u_1 + u_3/4$

$u_3 = \frac{1}{4}(0+0+u_2+u_4) = u_2 + 200/4$

$\therefore u_2 = \frac{1}{4}(u_2/4 + u_2 + 200/4)$

$\therefore u_2 = 100/7$

$u_1 = u_2/4 = 25/7$

$u_3 = u_2 + 200/4 = 375/7$

The final table is:

t,x	0.00	0.25	0.50	0.75	1.00
0	0	0	0	0	0
1	0	3.5714	14.2857	53.5714	200

4.b) Obtain the Laurent series which represent the function

$$f(z) = \frac{4z+3}{z(z-3)(z+2)} \text{ in the regions, i) } 2 < |z| < 3 \quad \text{ii) } |z| > 3 \qquad \textbf{(6)}$$

$f(z) = \frac{a}{z} + \frac{b}{z-3} + \frac{c}{z+2}$

$\therefore 4z + 3 = a(z-3)(z-2) + b\, z(z+2) + c\, z(z-3)$

When z = 0,   3 = -6a $\therefore$ a = -1/2

When z = -2,   -5 = 10c $\therefore$ c = -1/2

When z = 3,   15 = 15b $\therefore$ b = 1

$\therefore \dfrac{4z+3}{z(z-3)(z+2)} = \dfrac{-1}{2z} + \dfrac{1}{z-3} - \dfrac{1}{2(z+2)}$

When **2 < |z| < 3,**

$f(z) = \dfrac{-1}{2z} + \dfrac{1}{z-3} - \dfrac{1}{2(z+2)}$

$f(z) = \dfrac{-1}{2z} - \dfrac{1}{3\left(1-\frac{z}{3}\right)} - \dfrac{1}{2z(1+2/z)}$

$\qquad = \dfrac{-1}{2z} - \dfrac{1}{3}\left(1 + \dfrac{z}{3} + \dfrac{z^2}{3} + \cdots\right) - \dfrac{1}{2z}\left(1 + \dfrac{2}{z} + \dfrac{2^2}{z^2} + \ldots\right)$

When **|z| > 3**

$f(z) = \dfrac{-1}{2z} + \dfrac{1}{z-3} - \dfrac{1}{2(z+2)}$

$f(z) = \dfrac{-1}{2z} - \dfrac{1}{z\left(1-\frac{3}{z}\right)} - \dfrac{1}{2z(1+2/z)}$

$\qquad = \dfrac{-1}{2z} - \dfrac{1}{z}\left(1 + \dfrac{3}{z} + \dfrac{3^2}{z^2} \ldots\right) - \dfrac{1}{2z}\left(1 + \dfrac{2}{z} + \dfrac{2^2}{z^2} + \ldots\right)$

4.c) Solve $(D^2 - 3D + 2)\, y = 4e^{2i}$ with y (0) = -3 and y' (0) = 5 where

$D = \dfrac{d}{dt}$ **(8)**

$L(y) = \bar{y}$

Taking Laplace transform,

$L(y'') - 3L(y') + 2L(y) = a\, L(\mathbf{4e^{2i}})$

But, $L(y') = s\bar{y}\, y(0) = s\bar{y} + 3$

And $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2 y + 3s - 5$

∴ the equation becomes,

$$s^2 y + 3s - 5 - 3(s\overline{y} + 3) + 2\overline{y} = 4\frac{1}{s-2}$$

Solving both sides, we get

$$\overline{y} = -\frac{7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$$

Taking inverse Laplace transform,

$$y = -7\,L^{-1}\left(\frac{1}{s-1}\right) + 4L^{-1}\left(\frac{1}{s-2}\right) + 4L^{-1}\frac{4}{(s-2)^2}$$

$$= -7e^i\,L^{-1}\frac{1}{s} + 4\,e^{2i}L^{-1}\frac{1}{s^2}$$

$$\mathbf{= -7e^i + 4e^{2i} + 4ie^{2i}}$$

5.a) Find the bilinear transformation under which 1, i, -1 from the z plane are mapped onto $0, 1, \infty$ of w-plane. **(6)**

Let the transformation be $w = \frac{az+b}{cz+d}$

Putting the given values of z and w, we get,

$$0 = \frac{a+b}{c+d} \qquad 1 = \frac{ai+b}{ci+d} \qquad \infty = \frac{-a+b}{-c+d}$$

∴ a + b = 0   ∴ b = -a

∴ -c + d = 0   ∴ d = c

Now,

ai = b = ci + d   ∴ ai - a = ci + c   ∴ ai + b = ci + d

∴ ai – a = ci = c   a(i-1) = c(i + 1)

∴ $c = a\frac{i-1}{i+1}$ ∴ $c = a\frac{(i^2 - 2i + 1)}{i^2 - 1} = ai$

d = c = ai   ∴ $w = -i\frac{z-1}{z+1}$

Now, when $|w| = 1$, $\left| -i\frac{z-1}{z+1} \right| = 1$

∴ $|z-1| = |z+1|$

∴ |(x-1) -iy)| =|(x+1) + iy|

∴ $(x-1)^2 + y^2 = (x+1)^2 + y^2$

∴ 4x =0   ∴ **x =0**

Hence, the map is in y axis.

5.b) Find the Laplace transform of

f(t) = t, 0 < t < π

   π − t, π < t <2 π and f(t+2 π) = f(t)   **(6)**

Since, f(t) is periodic with period a = 2π, we have

$\text{Lf(t)} = \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t) dt$

$= \frac{1}{1-e^{-2\pi s}} [\int_0^\pi e^{-st}. t dt + \int_\pi^{2\pi} e^{-st}(\pi - t)dt$

Solving the above equation, we get

$$\text{Lf(t)} = \frac{1-(1+\pi s)e^{-\pi s}}{s^2(1+e^{-\pi s})}$$

5.c) Obtain half range Fourier cosine series of f(x) = x, 0 < x<2. Using Parseval's identity, deduce that −

$\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$ **(8)**

Let $f(x) = a_0 + \sum a_n \cos \frac{n\pi x}{l}$ here l=2.

∴ $a_0 = \frac{1}{l} \int_0^l f(x) dx = 1$

∴ $a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx$

Solving the above equation, we get,

$a_n = (2.(0) + \frac{\cos n\pi}{n^2\pi^2/2^2} - 0 - \frac{1}{n^2\pi^2/2^2}) = \frac{[(-1)^n - 1]}{n^2\pi^2/2^2}$

∴ $x = 1 - \frac{8}{\pi^2} [\frac{1}{1^2} \cos \frac{\pi x}{2} + \frac{1}{3^2} \cos \frac{3\pi x}{2} + \ldots\ldots]$

By Parseval's identity,

$\frac{1}{l} \int_0^l [f(x)]^2 dx = \frac{1}{2} [2a_0^2 + a_1^2 + a_2^2 + \ldots]$

$\therefore LHS = \frac{1}{l} \int_0^l [x]^2 dx = \frac{4}{3}$

$\therefore \frac{4}{3} = \frac{1}{2} [2 + \frac{64}{\pi^2} \{\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots\}]$

$\therefore \frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$

6.a) Using Contour Integration, evaluate:

$$\int_{-\infty}^{\infty} \frac{x^2+x+2}{x^4+10x^2+9} \, dx \qquad \qquad \textbf{(6)}$$

$f(x) = \dfrac{x^2+x+2}{x^4+10x^2+9}$

$f(z) = \dfrac{z^2+z+2}{z^4+10z^2+9}$

For poles, $z^4 + 10z^2 + 9 = 0$

Solving the equation, we get

$z = \pm i$ and $z = \pm 3i$

$z = i, 3i$ are the poles which lie inside the circle.

Residue of f(z) at z =i is:

$$= \lim_{z \to i}(z - i)[\frac{z^2+z+2}{z^4+10z^2+9}]$$

$$= \lim_{z \to i}(z - i)[\frac{z^2+z+2}{(z^2+9)(z^2+1)}]$$

$$= \lim_{z \to i}(z - i)[\frac{z^2+z+2}{(z^2+9)(z+i)(z-i)}]$$

$$= \frac{1+i}{16i}$$

Residue of f(z) at z =3i is:

$$= \lim_{z \to i}(z - i)[\frac{z^2+z+2}{z^4+10z^2+9}]$$

$$= \lim_{z \to i}(z - i)\left[\frac{z^2+z+2}{(z^2+9)(z^2+1)}\right]$$

$$= \lim_{z \to i}(z - i)\left[\frac{z^2+z+2}{(z-3i)(z+3i)(z^2+1)}\right]$$

$$= \frac{7-3i}{48i}$$

Sum of residues $:= \frac{5}{24i}$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2+x+2}{x^4+10x^2+9} = 2\pi i \cdot \frac{5}{24i} = \frac{5\pi}{12}$$

6.b) Using least square method, fit a parabola, y= a+bx+cx² to the following data, **(6)**

x	-2	-1	0	1	2
y	-3.150	-1.390	0.620	2.880	5.378

Sr. No	x	y	x²	x³	x⁴	xy	x²y
1	-2	-3.150	4	-8	16	6.300	-12.600
2	-1	-1.390	1	-1	1	1.390	-1.390
3	0	0.620	0	0	0	0.000	0.000
4	1	2.880	1	1	1	2.880	2.880
5	2	5.378	4	8	16	10.756	21.512
N = 5	0	4.338	10	0	34	21.326	10.402

The normal equations are

$\sum y = Na + b\sum x + c\sum x^2 \quad \therefore 4.338 = 5a + 10c$

$\sum xy = a\sum x + b\sum x^2 + c\sum x^3 \quad \therefore 21.325 = 10b$

$\sum x^2y = a\sum x^2 + b\sum x^3 + c\sum x^4 \quad \therefore 10.402 = 10a + 34c$

Solving these three equations, we get

a = 0.621, b =2.1326, c= 0.1233

Hence, the parabola is **y = 0.621 + 2.1326x + 0.1233x²**

6.c)Determine the solution of one dimensional heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ under the boundary conditions u(0,t) = 0, u(l,t) = 0, u(x,0) = x,(0< x < l), l being the length of the rod. **(8)**

The equation of heat flow $\frac{\partial u}{\partial t} = c^2 \frac{\partial u}{\partial t}$ .....(1)

Since the temperature of the two ends of the rod are zero, its solution is of the form

$$u = (c_1 \cos mx + c_2 \sin mx)e^{-m^2 c^2 t} \quad \ldots\ldots (2)$$

Using initial conditions,

(i) u =0 when x = 0, we get, from (2) $0 = c_1 e^{-m^2 c^2 t}$

$\therefore c_1 = 0$

$\therefore$ the solution (2) becomes $u = c_2 \sin mx \ e^{-m^2 c^2 t}$ .... (3)

(ii) u =0 when x=l, we get, from (3) $0 = c_2 \sin ml \ e^{-m^2 c^2 t}$

$$ml = n\pi \quad \text{therefore } m = \frac{n\pi}{l}$$

Hence, the general solution is $u = \sum b_n \sin \frac{n\pi x}{l} e^{-n^2 \pi^2 c^2 \frac{t}{l^2}}$ ....
(4)

(iii) When l = 0, u = x, we get from (4) $\sum b_n \sin \frac{n\pi x}{l}$ ... (5)

But this is a half-range Fourier sine series for the function f(x) = x in the range 0 to l.

$$b_n = \frac{2}{l} \int_0^l x \sin \frac{n\pi x}{l} dx = \frac{2}{l} \left[(x) \left(-\frac{l}{n\pi} \cos \frac{n\pi x}{l} \right) (1) \left(-\frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l} \right) \right]$$

$$= \frac{2}{l} \left[-\frac{l^2}{n\pi} \cos n\pi \right] = -\frac{2l}{n\pi} \cos n\pi, n = 1,2,3 \ldots\ldots \quad \ldots\ldots(6)$$

Hence, from (4) we get solution as

$$u = \frac{2l}{\pi} \left[\frac{1}{1} \sin \frac{\pi x}{l} e^{-\pi^2 c^2 \frac{t}{l^2}} - \frac{1}{2} \sin \frac{2\pi x}{l} e^{-4\pi^2 c^2 \frac{t}{l^2}} + \cdots \ldots \right]$$